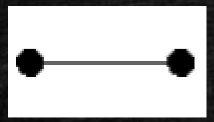
Basic Properties of Graphs

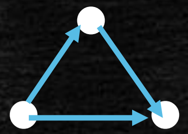
Unit 1

GRAPHS

Graph :

- It is a set of the form {(x,f(x)): x is a domain of function f}.

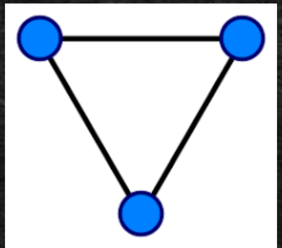




Each point is called a vertex. Line joining aby pair is called an edge. Edge from x_1 to x_2 is denoted by (x_1x_2)

$(x_1,x_2) \neq (x_2,x_1)$

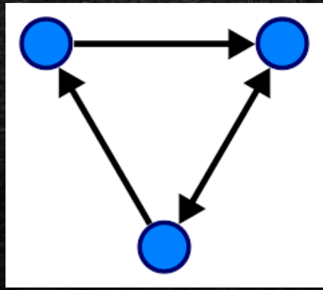
- Undirected graph G is a finite non-empty set V together with set E containing pairs of points of V. V is called the vertex set and E is the edge set of G. In undirected graph, E(G) will be symmetric on V(G). If (u,v) is there, then (v,u) will be there.



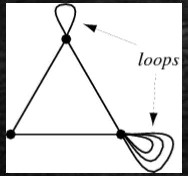
Relationship between V,E and G is :

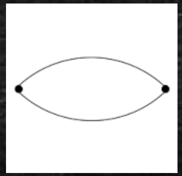
G= ((V(G)),(E(G)))

- Directed graph G is a finite non-empty set V together with subset E of Cartesian product of set V*V. In directed graph, E(G) will not be symmetric on V(G).

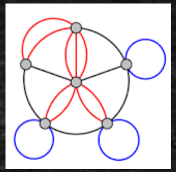


- Loop : When an edge joins a vertex to itself is called a loop.

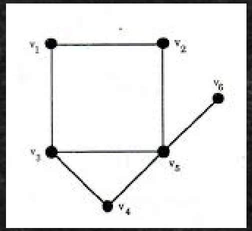




Parallel edges



Multigraph



Simple graph

- Parallel edges / multiple edges : Two or more edges join the same vertices.
- Multigraph : Graph that contains multiple edges is called a multigraph.
- Simple graph : Undirected graph that has no loops or multiple edges is called a simple graph.

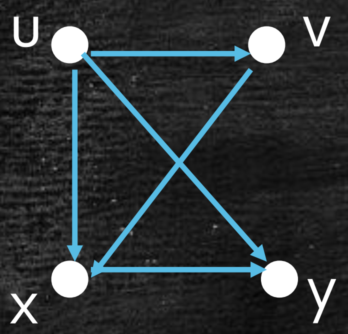
- If e is an edge joining vertices u and v, then :

1. u and v are adjacent vertices or neighbours.
2. u and v are the endpoints of e.
3. e is adjacent with u and v.

Adjacent edges : If distinct edges e_1 and e_2 have at least one vertex in common, then e_1 and e_2 are adjacent edges.

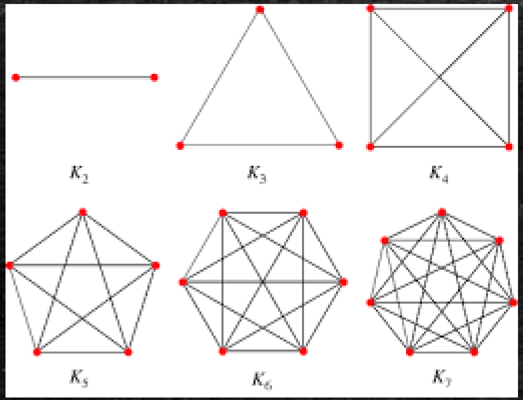
E.g. : G=(V,E)

Where V={u,v,x,y} E={uv,ux,uy,vx,xy}

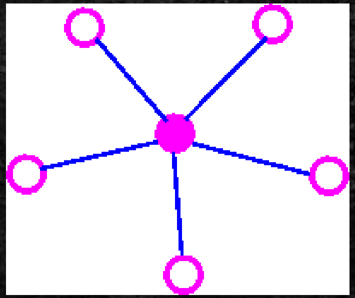


In the above figure,
v and y are non-adjacent vertices.
uv and vx are adjacent edges. V is a common vertices.
Non adjacent edges : uv and xy

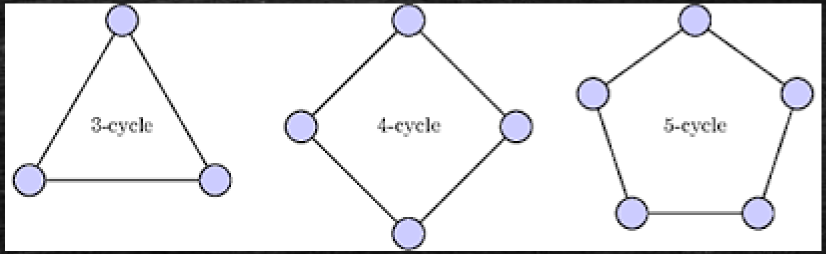
- Complete graph : Graph in which any two vertices are adjacent, i.e. each vertex is joined to every other vertex by a vertex. A complete graph on n vertices is represented by K_n.



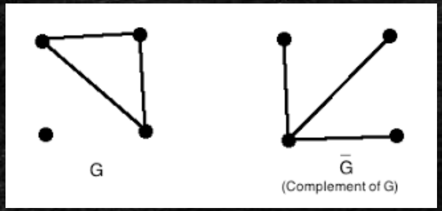
Star topology : In this graph, n vertices are adjacent to one central vertex.



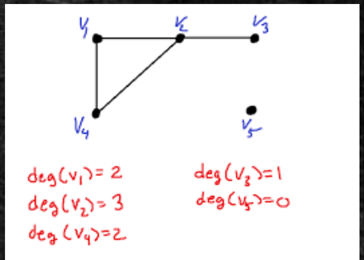
- Cycles: A cycle C_n is a graph on n vertices $\{x_1, \ldots, x_n\}$ where $E(C_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$.



- Complement of a graph : Let graph G=(V,E) be a (p,q) graph. Complement of the graph $\overline{G}$ is a graph $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy : xy \notin E(G), x, y \in V(G)\}$.



DEGREE, REGULARITY AND ISOMORPHISM



- Two vertices joined by an edge are called adjacent vertices or neighbours.
- The set of all neighbour of a vertex x of graph G is called the neighbourhood set of x. It is denoted by $N_G(x)$.
- Degree of a vertex x is the number of edges incident with x. It is denoted by $d_G(x)$.
- $d_G(x)=|N_G(x)|$, $|N_G(x)|$ is the number of elements of set $N_G(x)$.
- In a (p,q) graph G, the maximum number of edges incident with a vertex x is $0 \le d_G(x) \le (p-1)$ for vertex x in G.
- A vertex x of a graph G is called an **even vertex** if d_G (x) is even; otherwise it is called an **odd vertex**. A vertex with degree 0 is called an **isolated vertex**. In the above diagram, v1 and v4 are the even vertices, v3 and v2 are odd vertices and v5 is an isolated vertex.

Handshaking problem :

If G is a (p,q) graph with $V(G)=\{V_1....V_p\}$ and $d_i=d_G(V_i)$, $1 \le i \le p$, then $$2q = \sum_{i=1}^{p} d_i$$

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x \text{ is an endpoint of } e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

Corollary 1: Some of the degrees of all the vertices of a graph is even.

Proof : Consider a (p,q) graph with edgeset a subset of all set of all subsets of size of two elements of V(G).

$q \leq \ (p(p-1))/2$

According to theorem 1, there can't be a graph with vertices having given degree in all cases.

For eg: Consider a graph with 12 vertices, having 2 vertices with degree 1, 3 vertices with degree 3 and the remaining with degree 10.

$\sum d_i = 1+1+3+3+3+10+10+10+10+10+10+10=81$

Since $\sum d_i$ is not even, it does not satisfy theorem 1.

Corollary 2 : Any graph can only have an even number of odd vertices.

Consider a (p,q) graph with {x1,x2,.....xt} is a set of odd vertices and {xt+1,.....xp} is a set of even vertices.

Let $d_G(x_i)=2c_i+1 \quad 1 \le i \le t$ and $d_G(x_i)=2r_i \quad t+1 \le i \le p$

Then Theorem 1 says that $2q=\sum_{1}^{p} d_G(x_i)$

$$\Rightarrow 2q=\sum_{1}^{t}(2c_i+1)+\sum_{t+1}^{p}(2r_i) = 2(c_1+c_2+\Lambda+c_t)+t+2(r_{t+1}+\Lambda+r_p),$$

which shows that t is even.

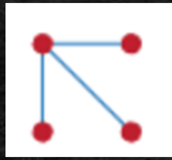
- Minimum vertex degree of a graph G :

δ (G)= min{d_G(x) :x∈ V(G)} is called the **minimum vertex degree of G,** and

Δ (G)=max{d_G(x) :x∈ V(G)} is called the **maximum vertex degree of G**

δ(G) and Δ(G) are non-negative integers.

Example :



δ(G)=1

Δ(G)=3

- Consider a (p,q) graph G, then degree sequence of graph is obtained by rearranging the vertices in decreasing order of their degrees.

Regular graph :

It is a graph in which each vertex has the same degree. It is said to be regular graph degree of regularity r. G is an r-regular graph where $0 \leq r \leq (p-1)$.

K_n is a regular graph with degree of regularity (n-1) when n > 3.

Isomorphic graphs :

Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

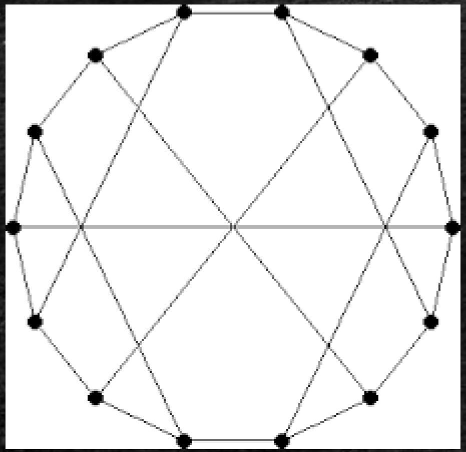
Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

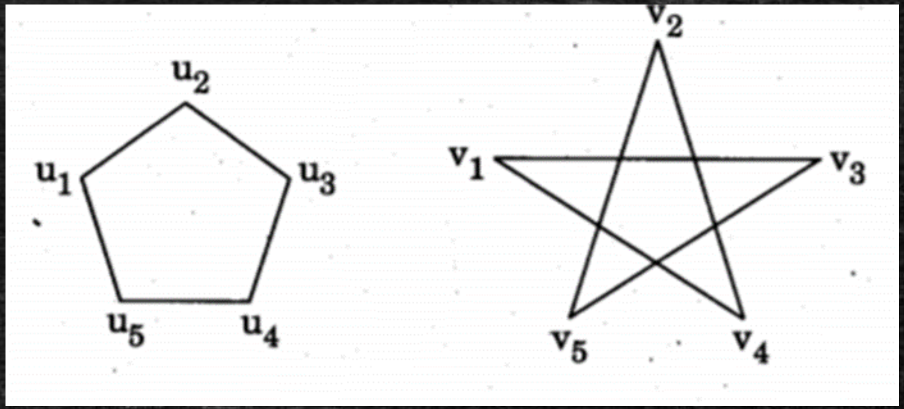
ii) $xy \in E(G)$ if and only if $f(x)\, f(y) \in E(H)$

If not they are called non-isomorphic graphs.

4-regular graph on 12 vertices



Isomorphic graph



To check if two graphs check for these conditions :

1. Count the number of vertices – must be equal
2. Count the number of edges – must be equal
3. Degree sequence – must be same
4. Number of cycles – must be same
5. Max degree vertex and min degree vertex
6. Peculiarity of adjacent vertices

Let f be an isomorphism from a graph G to a graph H. Then the following hold :

1. If G is a (p,q) graph then H must also be (p,q) graph.
2. The inverse map f^{-1} is an isomorphism from the graph H to the graph G.
3. Degree sequence of the graph G is the same as the degree sequence of the graph H.
4. For every positive integer $n \geq 3$, the number of copies of C_n in G is equal to the number copies of C_n in H.

Subgraphs

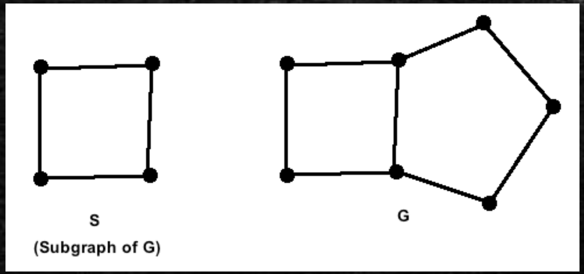
Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) ⊆ V (G) and E (H) ⊆ E (G).

If H is a subgraph of a graph G, such that V (H) = V (G) and E (H) ⊆ E (G), that is, H and G have exactly the same vertex set, then H is called a **spanning subgraph** of G.

Every graph G is a subgraph of itself, i.e., G is a subgraph of G.

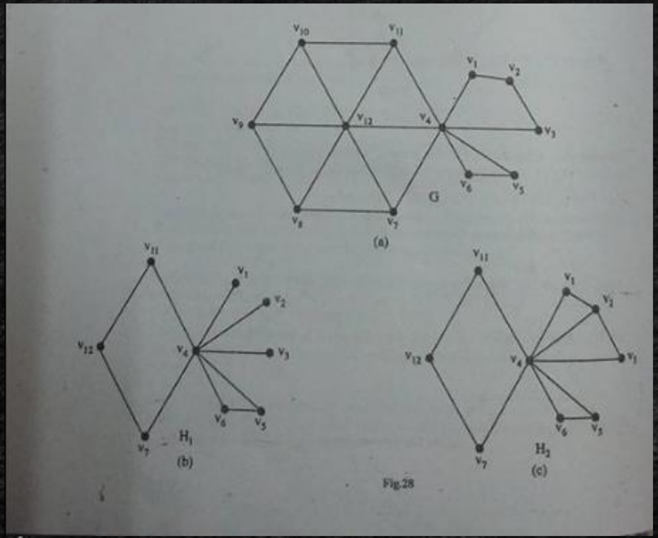
For any v ∈ V (G), {v} is a subgraph of G.

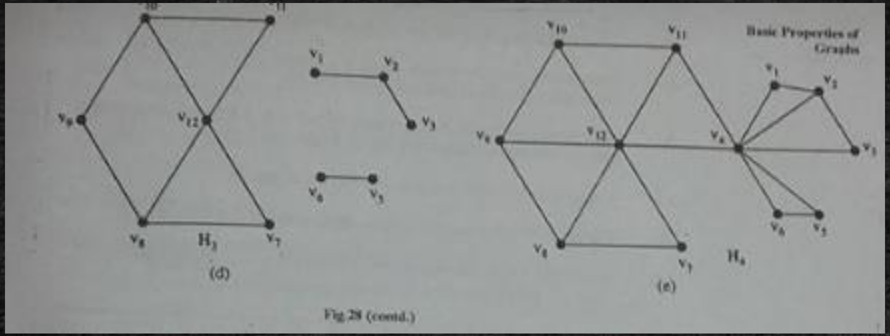
Let G be a graph and let S ⊆ V (G). By **the** subgraph of the graph G, induced by the set S, we mean the subgraph H with V(H)=S and the edge set consisting of those edges of G which are joining the vertices in S. That is, E(H)= {xy:x≠y,x ∈ S,y ∈ S,xy ∈ E(G)}= We denote H by $<S>_G$.



Note that for a vertex $v \in V(G)$, by $G - v$ we mean the subgraph $< V(G) - \{v\} >_G$, which means a subgraph of G consisting of all points of G except v, and all edges of G except for the edges incident with v.

For a subset S of V (G), the subgraph $< V(G) - S >_G$ is often written as $G - S$.





In the above figure,

a: a graph G

b: subgraph H_1

c:a vertex induced subgraph H_2 with $V(H_2)=V(H_1)$

d:$H3=G-v_4$

e: spanning subgraph H4